

# Unstable Particles<sup>1</sup>

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## Abstract

Unstable particles cannot be treated as asymptotic external states in  $S$ -matrix theory and when they occur as resonant states cannot be described by finite-order perturbation theory. The known facts concerning unstable particles are reviewed and it is shown how to construct gauge-invariant expressions for matrix elements containing intermediate unstable particles and physically meaningful production cross-sections for unstable particles. The results and methodology presented are relevant for  $Z^0$  resonance physics,  $W^+W^-$  and  $Z^0Z^0$  pair production and can be straightforwardly applied to other processes.

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# 1 Introduction

It is a fact of life that the heaviest known elementary particles, the  $W$  and  $Z^0$  bosons and the top quark have total decay widths that are a substantial fraction of their masses. Yet  $S$ -matrix theory does not easily deal with unstable particles as they cannot be represented by asymptotic states. The study of width effects is timely and important for the theoretical understanding of current and future experimental results from SLC, LEP and hadron colliders. The unstable  $Z^0$  boson has by now been produced in vast numbers at SLC and LEP and the latter machine is soon begin producing  $W^+W^-$  pairs. Width effects have been suggested as a possible mechanism for generating enhanced CP-violating observables that could provide a window into new physics.[1, 2] A number of attempts have been made to treat width effects by introducing an imaginary part by hand into the propagator of the unstable particle[3, 4, 5, 6]. Such approaches wreak havoc with unitarity and current conservation and must then be patched in some manner. In patching it is, in some cases, found necessary to introduce an imaginary part into propagators of unstable particles in the  $t$ -channel which is obviously incorrect. It is not easy to extend such methods to higher orders. A comparison of the relative merits of various approaches appears in ref.s [7] and [8]. The analysis presented here suffers from none of these difficulties and is fully consistent with constraints imposed by analytic  $S$ -matrix theory and perturbation theory. Much of what appears here can be found in ref.s [9, 10, 11] and [12].

In this paper we consider two distinct reactions involving unstable particles. The first is  $e^+e^- \rightarrow f\bar{f}$  near the  $Z^0$  resonance with  $f$  used to denote a generic light fermion species. This reaction proceeds via two distinct mechanisms. The annihilation can produce a  $Z^0$  boson that subsequently decays into the  $f\bar{f}$  pair or they can be produced directly without the intermediate production of a  $Z^0$  boson. For this process it is not too difficult to treat the reaction in terms of the stable external fermions. The  $Z^0$  has its 4-momentum fixed by the incoming  $e^+e^-$  and therefore does not participate in phase space integrals. Nevertheless, to account for the Breit-Wigner resonance shape, some sort of resummation of the perturbation expansion is required. Done carelessly and without due regard to the constraints and requirements imposed by analytic  $S$ -matrix theory, the resummation can lead to manifestly gauge-dependent results. The solution to the problem is shown to be to perform a Laurent expansion on the complete matrix element about the pole. This naturally generates exactly gauge-invariant expressions. An important point to note is that the Laurent expansion is much more than a mathematical trick. There is physics in the expansion. The leading term is the one responsible for the Breit-Wigner resonance structure and describes the production, propagation and subsequent decay of a physical unstable particle, in this case the  $Z^0$ . The remaining non-resonant background accounts for the prompt production of the final state fermions. As these processes are physically distinguishable in principle it is essential that their corresponding contributions to the matrix element be maintained separate and not combined in any way. Thus a Laurent

expansion should be performed even when using a perturbation expansion, such as the background field method[13], that generates gauge-invariant Green functions.

The second process that will be considered is the process  $e^+e^- \rightarrow (f_1\bar{f}_1)(f_2\bar{f}_2)$  for energies above the  $Z^0Z^0$  production threshold. Here a complete treatment of the matrix element in terms of the six external stable fermions becomes unwieldy. The calculation would be simplified by considering just  $e^+e^- \rightarrow Z^0Z^0$ , that is expected to be the dominant source of  $(f_1\bar{f}_1)(f_2\bar{f}_2)$ , but  $S$ -matrix theory cannot tolerate unstable particles as external states. Also the invariant mass of the  $Z^0$ 's is not fixed but varies during phase-space integrations. It will be shown, however, that from the matrix element for  $e^+e^- \rightarrow (f_1\bar{f}_1)(f_2\bar{f}_2)$ , it is possible to isolate a piece that describes propagation of  $Z^0$ 's over a finite-range. This piece when suitably squared and summed over final state fermions yields a physically meaningful cross-section for the process  $e^+e^- \rightarrow Z^0Z^0$ . As such it must be exactly gauge-invariant. It turns that the required part of the matrix element is precisely that which is obtained by a Laurent expansion in the invariant masses of the  $Z^0$  bosons.

## 2 Properties of Unstable Particles

We begin by recalling what is known about unstable particles.  $S$ -matrix theory tells us that unstable particles in intermediate states are associated with poles in their invariant momentum,  $s$ , in the  $S$ -matrix element lying off the physical sheet below the real  $s$ -axis. The residue of the  $S$ -matrix element factorizes as a consequence of Fredholm theory (see ref. [14], p. 253). These residues can be used to define generalized  $S$ -matrix elements for processes with unstable particles as external states. These generalized  $S$ -matrix elements satisfy unitarity relations that are analogous to those for stable particles but continued off the real axis[15, 16]. Unfortunately the unitarity relations no longer relate real quantities to one and other and it is unclear what they have to do with physically measurable cross-sections.

It is known that the lifetime of an unstable particle depends on how it was prepared [17]. The lifetime is therefore an ill-defined concept without specifying further information such as that the unstable particle is in a state of definite 4-momentum.

Veltman [18] showed that in models containing unstable particles the  $S$ -matrix was unitary and causal on the Hilbert space spanned by stable particle states. That is to say that only stable states must be included in unitarity summations. Hence there is not really even room to accommodate unstable particles as external states.

For an excellent review of some of the properties of unstable particles see ref. [19].

## 3 Gauge Invariance near Resonance

With the advent of LEP, the unstable  $Z^0$  has been produced in large numbers. The physics is described by the Standard Model Lagrangian with which calculations can

be done to arbitrary order in perturbation theory. The catch is that the  $Z^0$  resonance is a fundamentally non-perturbative object and, in order to account for the resonance structure, some sort of Dyson summation must be performed. As first pointed out in ref. [3], the resummation can lead to a breaking of gauge-invariance. To see how this happens consider the process  $e^+e^- \rightarrow f\bar{f}$  far from the  $Z^0$  resonance, say at PETRA energies of around 30 GeV. There perturbation theory works without difficulties as a resummation need not and should not be performed. The results generated by the perturbation expansion are exactly gauge-invariant in each order. Let us look at how the exact gauge-invariance is realized. In what follows we will use  $\Pi_{ZZ}^{(1)}(q^2)$  to denote the transverse part of the  $Z^0$  boson one-loop self-energy. The superscript in parentheses indicates the loop order. The absence of a superscript will be taken to indicate the exact expression to all orders and  $^{(0)}$  used for vertices indicates the tree-level. The initial state vertex correction that connects the initial-state electron line to the  $Z^0$  is given by  $V_{iZ}(q^2)$  and the final-state vertex connecting the  $Z^0$  to the final-state fermion line by  $V_{Zf}(q^2)$ . Although we will initially ignore the existence of the photon, the transverse parts of the photon self-energy and the  $Z$ - $\gamma$  mixing are  $\Pi_{\gamma\gamma}(q^2)$  and  $\Pi_{Z\gamma}(q^2)$  and respectively.  $V_{i\gamma}(q^2)$  and  $V_{\gamma f}(q^2)$  are the initial- and final-state photon vertices.  $B(s, t)$  denotes one particle irreducible (1PI) corrections, to the matrix element. These include things like as box diagrams.

For the process  $e^+e^- \rightarrow f\bar{f}$  tree-level plus one-loop corrections to the scattering amplitude read

$$A(s, t) = \frac{V_{iZ}^{(0)} V_{Zf}^{(0)}}{(s - M_Z^2)} + \frac{V_{iZ}^{(0)} \Pi_{ZZ}^{(1)}(s) V_{Zf}^{(0)}}{(s - M_Z^2)^2} + \frac{V_{iZ}^{(1)}(s) V_{Zf}^{(0)}}{(s - M_Z^2)} + \frac{V_{iZ}^{(0)} V_{Zf}^{(1)}(s)}{(s - M_Z^2)} + B^{(1)}(s, t) \quad (1)$$

that is exactly gauge-invariant. Here  $M_Z$  is the unphysical renormalized  $Z^0$  boson mass. The correction (1) can be split into separately gauge-invariant pieces by classifying the various contributions according to their pole structure. Writing

$$\frac{\Pi_{ZZ}^{(1)}(s)}{(s - M_Z^2)^2} = \frac{\Pi_{ZZ}^{(1)}(M_Z^2)}{(s - M_Z^2)^2} + \frac{\Pi_{ZZ}^{(1)'}(M_Z^2)}{(s - M_Z^2)} + \Pi_{ZZ}^{(1)R}(s) \quad (2)$$

$$\frac{V_{iZ, Zf}^{(1)}(s)}{(s - M_Z^2)} = \frac{V_{iZ, Zf}^{(1)}(M_Z^2)}{(s - M_Z^2)} + V_{iZ, Zf}^{(1)R}(s) \quad (3)$$

the coefficients of the pieces of (1) having a double, single and no pole at  $s = M_Z^2$  are

$$\Pi_{ZZ}^{(1)}(M_Z^2), \quad (4)$$

$$V_{iZ}^{(0)} \Pi_{ZZ}^{(1)'}(M_Z^2) V_{Zf}^{(0)} + V_{iZ}^{(1)}(M_Z^2) V_{Zf}^{(0)} + V_{iZ}^{(0)} V_{Zf}^{(1)}(M_Z^2), \quad (5)$$

$$V_{iZ}^{(0)} \Pi_{ZZ}^{(1)R}(s) V_{Zf}^{(0)} + V_{iZ}^{(1)R}(s) V_{Zf}^{(0)} + V_{iZ}^{(0)} V_{Zf}^{(1)R}(s) + B^{(1)}(s, t), \quad (6)$$

respectively. Since Eq. (1) is exactly gauge-invariant and since the cancellation of gauge-dependence cannot occur between the various terms of differing pole structure

we must conclude that (4)–(6) are separately and exactly gauge-invariant. We will find that these combinations will reappear in the treatment of the  $Z^0$  resonance. While Eq. (1) is gauge-invariant it blows up hopelessly as  $s \rightarrow M_Z^2$  and therefore cannot be used to treat the  $Z^0$  resonance. What is normally done is to perform a Dyson summation of the  $Z^0$  self-energy corrections to obtain an expression for the matrix element near resonance

$$A(s, t) = \frac{V_{iZ}^{(0)} V_{Zf}^{(0)} + V_{iZ}^{(1)}(s) V_{Zf}^{(0)} + V_{iZ}^{(0)} V_{Zf}^{(1)}(s)}{s - M_Z^2 - \Pi_{ZZ}^{(1)}(s)} + B^{(1)}(s, t). \quad (7)$$

The problem, as pointed out in ref. [3], is that  $A(s, t)$  is now gauge-dependent. The reason is that the Dyson summation has included all corrections of the form  $\left(\frac{\Pi_{ZZ}^{(1)}(s)}{s - M_Z^2}\right)^n$  each of which is gauge-dependent. In the complete matrix element, the gauge-dependence of these terms would be canceled by combinations of higher order self-energy corrections,  $\Pi_{ZZ}^{(n)}(s)$ , vertex corrections,  $V_{iZ, Zf}^{(n)}(s)$ , and 1PI corrections,  $B^{(n)}(s, t)$ . None of these are present in Eq. (7) and hence it is gauge-dependent. Far from resonance the gauge-dependence starts formally at  $\mathcal{O}(\alpha^2)$  compared to the tree-level result but near the resonance where the leading  $s - M_Z^2$  in resonant denominator of Eq. (7) becomes small and the gauge-dependence starts at  $\mathcal{O}(\alpha)$  compared to the lowest order result.

What is the solution to this problem? One that has become popular is to construct self-energy and vertex corrections that are gauge-invariant by themselves so that the resummations cannot generate spurious gauge-dependence[20, 21, 22]. But this is only symptomatic relief for a much deeper illness which the approach fails to address. In fact patching things in this way may generate unforeseen and undesirable side-effects such as shifting the pole or residue[9].

When  $e^+e^-$  annihilate at energies near the  $Z^0$  resonance they produce a physical  $Z^0$  that endures a while and then decays, normally to a fermion-antifermion pair,  $f\bar{f}$ . In a *gedanken* world where experimental resolution is extremely high or where the couplings of the  $Z^0$  are extremely weak, the presence of the  $Z^0$  could be detected as a physical particle in the way that neutrons, kaons and muons are. When  $e^+e^-$  annihilate they can also produce the final state  $f\bar{f}$  without producing a propagating  $Z^0$ . In this case no  $Z^0$  would be detected no matter how good the experimental resolution or how weak the couplings are made. Of course, there will always be a proportion of  $Z^0$  that will decay below the limit of experimental resolution.

The  $f\bar{f}$  is produced by two distinguishable mechanisms and thus the matrix element must always separate into the two corresponding pieces. How can these two pieces be identified in the expression for the matrix element? Consider the dressed propagator, for simplicity, of an unstable scalar particle. In coordinate space it is

$$\Delta(x' - x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik \cdot (x' - x)}}{k^2 - M^2 - \Pi(k^2) + i\epsilon} \quad (8)$$

The integrand has a pole at  $s = s_p$  that is a solution of the equation  $s_p - M^2 - \Pi(s_p) = 0$ . The denominator of the integrand may be written

$$k^2 - M^2 - \Pi(k^2) = \frac{k^2 - s_p}{F(k^2)} \quad (9)$$

The function  $F(k^2)$  is analytic and  $F(s_p) = (1 + \Pi'(s_p))^{-1}$ . Assuming  $t \neq t'$  and writing Eq. (8) in the form

$$\Delta(x' - x) = \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x' - x)} \left[ \frac{F(s_p)}{k^2 - s_p} + \frac{F(k^2) - F(s_p)}{k^2 - s_p} \right]. \quad (10)$$

integration with respect to  $k_0$  yields

$$\Delta(x' - x) = -i \int \frac{d^3 k}{(2\pi)^3 2k_0} [e^{-ik \cdot (x' - x)} \theta(t' - t) + e^{ik \cdot (x' - x)} \theta(t - t')] F(s_p). \quad (11)$$

The first term in Eq. (10) has split into two pieces and the second has vanished because it has no poles. As both terms in Eq. (10) are Lorentz invariant the vanishing of the second in all reference frames for  $t \neq t'$  labels it as a contact interaction. Similarly the first term is identifiable as an interaction of finite space-time range. It has a pole at  $k^2 = s_p$  and is the leading term in the Laurent expansion of the integrand about  $s_p$ .

This insight can now be applied to our generic process  $e^+e^- \rightarrow f\bar{f}$ . The exact scattering amplitude to all orders in perturbation theory takes the form

$$A(s, t) = \frac{V_{iZ}(s)V_{Zf}(s)}{s - M_Z^2 - \Pi_{ZZ}(s)} + B(s, t). \quad (12)$$

The right-hand side has a pole at  $s = s_p$  and making a Laurent gives

$$A(s, t) = \frac{V_{iZ}(s_p)F_{ZZ}(s_p)V_{Zf}(s_p)}{s - s_p} + \frac{V_{iZ}(s)F_{ZZ}(s)V_{Zf}(s) - V_{iZ}(s_p)F_{ZZ}(s_p)V_{Zf}(s_p)}{s - s_p} + B(s, t). \quad (13)$$

where  $F_{ZZ}(s)$  is defined through the relation (9). It may be shown [10] that the pole position, residue and background are separately and exactly gauge-invariant. They may therefore be separately expanded as a perturbation series about  $M_Z^2$  giving

$$\begin{aligned} A(s, t) = & \frac{V_{iZ}^{(0)}V_{Zf}^{(0)} + V_{iZ}^{(0)}\Pi_{ZZ}^{(1)'}(M_Z^2)V_{Zf}^{(0)} + V_{iZ}^{(1)}(M_Z^2)V_{Zf}^{(0)} + V_{iZ}^{(0)}V_{Zf}^{(1)}(M_Z^2)}{s - s_p} \\ & + V_{iZ}^{(0)}\Pi_{ZZ}^{(1)R}(s)V_{Zf}^{(0)} + V_{iZ}^{(1)R}(s)V_{Zf}^{(0)} + V_{iZ}^{(0)}V_{Zf}^{(1)R}(s) + B^{(1)}(s, t) \end{aligned} \quad (14)$$

with, to  $\mathcal{O}(\alpha^2)$ ,

$$s_p = M_Z^2 + \Pi_{ZZ}^{(1)}(M_Z^2) + \Pi_{ZZ}^{(2)}(M_Z^2) + \Pi_{ZZ}^{(1)}(M_Z^2)\Pi_{ZZ}^{(1)'}(M_Z^2). \quad (15)$$

Note that the pole position, residue and background are precisely the gauge-invariant combinations identified in (4)–(6). The overall result Eq. (14) is clearly gauge-invariant and naturally separates into finite range and contact interaction as expected. As stated above the resonant finite range piece is associated with the production of a physical  $Z^0$  and the contact interaction is associated with prompt production of the final-state fermions via non-propagating modes of the  $Z^0$  field and box diagrams.

It should be emphasized that nowhere in this derivation did we introduce an *ad hoc* width by hand as has been done by a number of authors [3, 4, 5, 6]. The finite width appeared naturally in the Laurent expansion about  $s_p$ . The subsequent expansion of the pole position, residue and background about the renormalized mass,  $M_Z$ , is justified because these quantities represent three independent physical observables[10]. Since the procedure represents a well-defined sequence of expansions applied to the complete matrix element, the result can be automatically guaranteed not to violate unitarity or gauge-invariance.

Up to now the photon has been left out of the analysis. The Dyson summation will clearly be complicated by the addition of photon exchange diagrams and by  $Z$ - $\gamma$  mixing. Baulieu and Coquereaux [23] have shown how to perform the summation of the transverse parts exactly. The result is that the full matrix element for  $e^+e^- \rightarrow f\bar{f}$  is

$$\begin{aligned}
A(s, t) = & V_{i\gamma}(s) \frac{s - M_Z^2 - \Pi_{ZZ}(s)}{[s - \Pi_{\gamma\gamma}(s)] [s - M_Z^2 - \Pi_{ZZ}(s)] - \Pi_{\gamma Z}^2(s)} V_{\gamma f}(s) \\
& + V_{i\gamma}(s) \frac{\Pi_{\gamma Z}(s)}{[s - \Pi_{\gamma\gamma}(s)] [s - M_Z^2 - \Pi_{ZZ}(s)] - \Pi_{\gamma Z}^2(s)} V_{Zf}(s) \\
& + V_{iZ}(s) \frac{\Pi_{\gamma Z}(s)}{[s - \Pi_{\gamma\gamma}(s)] [s - M_Z^2 - \Pi_{ZZ}(s)] - \Pi_{\gamma Z}^2(s)} V_{\gamma f}(s) \\
& + V_{iZ}(s) \frac{s - \Pi_{\gamma\gamma}(s)}{[s - \Pi_{\gamma\gamma}(s)] [s - M_Z^2 - \Pi_{ZZ}(s)] - \Pi_{\gamma Z}^2(s)} V_{Zf}(s) \\
& + B(s, t)
\end{aligned} \tag{16}$$

exactly. The first term, representing photon exchange, can be split into a resonant and non-resonant piece. Collecting the resonant pieces together yields

$$\begin{aligned}
A(s, t) = & \frac{\left[ V_{i\gamma}(s) \frac{\Pi_{\gamma Z}(s)}{s - \Pi_{\gamma\gamma}(s)} + V_{iZ}(s) \right] \left[ V_{Zf}(s) + \frac{\Pi_{\gamma Z}(s)}{s - \Pi_{\gamma\gamma}(s)} V_{\gamma f}(s) \right]}{s - M_0^2 - \Pi_{ZZ}(s) - \frac{\Pi_{\gamma Z}^2(s)}{s - \Pi_{\gamma\gamma}(s)}} \\
& + \frac{V_{i\gamma}(s) V_{\gamma f}(s)}{s - \Pi_{\gamma\gamma}(s)} + B(s, t).
\end{aligned} \tag{17}$$

This exact scattering amplitude has a pole at the point,  $s_p$ , satisfying the equation

$$s_p - M_Z^2 - \Pi_{ZZ}(s_p) - \frac{\Pi_{\gamma Z}^2(s_p)}{s_p - \Pi_{\gamma\gamma}(s_p)} = 0. \tag{18}$$

Defining the function  $F_{ZZ}(s_p)$  from the relation

$$s - M_Z^2 - \Pi_{ZZ}(s) - \frac{\Pi_{Z\gamma}^2(s)}{s - \Pi_{\gamma\gamma}(s)} = \frac{1}{F_{ZZ}(s)}(s - s_p) \quad (19)$$

and extracting the leading term of Eq. (17) in the Laurent expansion about  $s_p$  leads to

$$\begin{aligned} A(s, t) &= \frac{R_{iZ}(s_p)R_{Zf}(s_p)}{s - s_p} \\ &+ \frac{R_{iZ}(s)R_{Zf}(s) - R_{iZ}(s_p)R_{Zf}(s_p)}{s - s_p} + \frac{V_{i\gamma}(s)V_{\gamma f}(s)}{s - \Pi_{\gamma\gamma}(s)} + B(s, t) \end{aligned} \quad (20)$$

in which

$$\begin{aligned} R_{iZ}(s) &= \left[ V_{i\gamma}(s) \frac{\Pi_{\gamma Z}(s)}{s - \Pi_{\gamma\gamma}(s)} + V_{iZ}(s) \right] F_{ZZ}^{\frac{1}{2}}(s), \\ R_{Zf}(s) &= F_{ZZ}^{\frac{1}{2}}(s) \left[ V_{Zf}(s) + \frac{\Pi_{Z\gamma}(s)}{s - \Pi_{\gamma\gamma}(s)} V_{\gamma f}(s) \right]. \end{aligned}$$

The equation (17) is **exact**. It is remarkably simple in structure; much more so than many approximate formulas that have appeared in the literature. The factorization of the residue at the pole, demanded by analytic  $S$ -matrix theory is manifest. The photon exchange contribution appears in the background in a transparent form. As with Eq. (14) the pole position, residue and background can be expanded separately as well-behaved perturbation expansions about the renormalized mass  $M_Z^2$ . The background is regular in  $s$  and can be expanded as a Taylor series. The Laurent expansion has separated the finite range and contact interaction contributions to the matrix element and should be performed even in the case where gauge-invariant self-energies are produced in a consistent manner such as with the background field method[13]. Expanding in the way shown here, however, makes the necessity or advantage of gauge-invariant self-energies unclear.

The results obtained in this section have been shown to be consistent with Ward identities[24].

## 4 The Mass and Width of an Unstable Particle

The quantity  $M_Z$  in the formulas given in the previous section is the renormalized mass of the  $Z^0$  boson in the scheme dictated by the counterterms contained in the self-energies and vertex corrections. Expressions for the counterterms appearing in the pole position, residue and background in the Standard Model for a general renormalization scheme can be found in ref. [10].



The renormalized mass is merely a bookkeeping device that has no physical content. This is clear from the fact that, in the  $\overline{\text{MS}}$  renormalization scheme, the renormalized mass depends on the arbitrary scale,  $\mu$ , and in the on-shell scheme it is gauge-dependent[25, 26] as it is entitled to be. The physical quantity associated with any particle is the position of the pole of its dressed propagator. For a stable particle the pole position is real and it may be identified with the physical mass of the particle. The renormalized mass may be set equal to this physical mass by appropriate choice of counterterms.

For an unstable particle the pole position is complex. It is the pole position as a whole that is the physically meaningful entity. The real and imaginary parts taken separately have no physical significance as they never appear separately in any expression for a physical observable. They are no more meaningful than, say, the modulus and argument of the pole position. Traditionally[27, 28] for convenience the pole was decomposed in two possible ways,

$$s_p = M_Z^2 - i\Gamma_Z M_Z \quad (21)$$

or

$$s_p = \left( M_Z - i\frac{\Gamma_Z}{2} \right)^2. \quad (22)$$

Both definitions are arbitrary. It was pointed out independently in ref.s [9] and [29] that the traditional definition of the mass lies below the on-shell renormalized mass, ostensibly being extracted by LEP, by 34 MeV in the case of the definition (21) and 26 MeV in the case of (22). Sirlin[25] modified the definition of the renormalized mass in the on-shell scheme to be just  $(M_Z^{\text{OS}})^2 = M_Z^2 - \Gamma_Z^2$  with  $M_Z$  and  $\Gamma_Z$  being taken from (21). It is not clear however that this can play the rôle of a self-consistent renormalized mass without violating Ward identities. Further discussion of these points can be found in ref. [11] where it is also suggested that the residue factors at the pole may be used to define model-independent partial widths.

At this point a word is appropriate about the range of validity of the Laurent expansion used in the previous section. It is correct up to a radius determined by the nearest singularity to the pole,  $s_p$ . If there are thresholds in the resonance region then these originate branch cuts and the expansion breaks down. However some interesting physics then appears. In fact for the  $Z^0$ , there are a multitude thresholds separated, at most by the mass of a light fermion pair, lying under the resonance. For example,  $Z^0 \rightarrow W^+ b \bar{b} s \bar{c}$ ,  $Z^0 \rightarrow 10(b \bar{b})$  lie under the umbrella resonance but are sufficiently weak as make them entirely negligible. Suppose for a moment that the top quark had had a mass that was  $m_t \approx M_Z/2$  then the first-order  $t \bar{t}$  threshold would lie under the peak of the resonance. The resonance region would be under the influence of two distinct poles, one reached by crossing the real  $s$ -axis onto the unphysical sheet below the threshold branch cut and the other by crossing above. Bhattacharya and Willenbrock[30] have examined this scenario in a toy model and found that the resonance peak becomes asymmetrical being narrower on the low energy side which

is easily understood in that the decay width increases as the threshold is crossed. In the situation where  $m_t \approx M_Z/2$  both poles must be associated with the  $Z^0$  boson. The  $Z^0$  resonance then becomes a closely-spaced doublet with a separation of roughly 100 MeV [11].

## 5 Production Cross-sections for Unstable Particles

In the foregoing we have treated the complete process  $e^+e^- \rightarrow f\bar{f}$ . This process possesses a number of simplifying features. There are only a total of four external particles and the 4-momentum of the intermediate unstable  $Z^0$  is fixed by the incoming  $e^+e^-$ . In contrast the process  $e^+e^- \rightarrow (f_1\bar{f}_1)(f_2\bar{f}_2)$  has an total of six external particles and therefore is much more complicated to deal with especially if radiative corrections are to be computed. It would be advantageous if the process  $e^+e^- \rightarrow Z^0 Z^0$  could be treated in some meaningful way as it is expected to be the dominant mechanism operating in the production of the  $(f_1\bar{f}_1)(f_2\bar{f}_2)$ . Now instead of single unstable particle there are two and their 4-momenta are no longer fixed but must be integrated over in phase-space integrations. The blind application of kinematic identities can generate complex scattering angles because of the presence of imaginary parts in particle self-energies. Some care is therefore required.

As stated earlier  $S$ -matrix theory cannot treat unstable particles as asymptotic states but as shown above it is possible to isolate the pieces of the matrix element for  $e^+e^- \rightarrow (f_1\bar{f}_1)(f_2\bar{f}_2)$  that correspond to the production, propagation and subsequent decay of physical  $Z^0$  bosons by means of the Laurent expansion. Summing over all possible stable decay products then gives the total production cross-section for the unstable particle.

Considering  $e^+e^- \rightarrow Z^0 Z^0$  avoids the need to confront the the issue of the Coulomb singularity that is present in  $e^+e^- \rightarrow W^+W^-$  when the  $W^+W^-$  pair are produced at low relative velocity[31, 32, 33]. At lowest order there are no Feynman diagrams containing the triple vector boson vertex.

With  $u$  and  $v$ , as usual, used to denote the spinor wavefunctions of the external fermions, the part of the full matrix element that can generate finite-range interactions is

$$\begin{aligned} \mathcal{M} = \sum_i [\bar{v}_{e^+} T_{\mu\nu}^i u_{e^-}] M_i(s, t, u, p_1^2, p_2^2) &\times \frac{1}{p_1^2 - M_Z^2 - \Pi_{ZZ}^2(p_1^2)} [\bar{u}_{f_1} V_{Zf_1}^\mu(p_1^2) v_{\bar{f}_1}] \\ &\times \frac{1}{p_2^2 - M_Z^2 - \Pi_{ZZ}^2(p_2^2)} [\bar{u}_{f_2} V_{Zf_2}^\nu(p_2^2) v_{\bar{f}_2}] \end{aligned} \quad (23)$$

where  $T_{\mu\nu}^i$  are kinematic tensors that span the tensor structure of the matrix element and the  $M_i$  are the associated form factors that are analytic in their arguments. It may be convenient, although not at all necessary, to construct the  $T_{\mu\nu}^i$  so as to be

individually gauge-invariant [34].  $M_i$  is a function of the usual Mandelstam variables  $s$ ,  $t$  and  $u$  and of the invariant masses  $p_1^2$  of the  $f_1\bar{f}_1$  pair and  $p_2^2$  of the  $f_2\bar{f}_2$ .

The expression (23) has been obtained by Dyson summation of the  $Z^0$  self-energies. It is not by itself gauge-invariant as it still contains contact interaction terms that must be split off. In fact the complete matrix element for  $e^+e^- \rightarrow (f_1\bar{f}_1)(f_2\bar{f}_2)$  naturally separates into four distinguishable processes according to whether not it is resonant or not in  $p_1^2$  or  $p_2^2$ . Fredholm theory guarantees the exact factorization of final-state contributions in the resonant channels. The four distinguishable physical processes are

$$\begin{aligned} e^+e^- &\longrightarrow \begin{cases} Z^0 \rightarrow f_1\bar{f}_1 \\ Z^0 \rightarrow f_2\bar{f}_2 \end{cases} \\ e^+e^- &\longrightarrow \begin{cases} Z^0 \rightarrow f_1\bar{f}_1 \\ f_2\bar{f}_2 \end{cases} \\ e^+e^- &\longrightarrow \begin{cases} f_1\bar{f}_1 \\ Z^0 \rightarrow f_2\bar{f}_2 \end{cases} \\ e^+e^- &\longrightarrow \begin{cases} f_1\bar{f}_1 \\ f_2\bar{f}_2 \end{cases} \end{aligned}$$

As for  $e^+e^- \rightarrow f\bar{f}$  they become physically distinguishable in the *gedanken* world of very high detector resolution or very weak couplings.

As it stands Eq. (23) contains the entire contribution of the matrix element for the first process listed above and parts of the remaining three. Additional Feynman diagrams must be included to compute the full matrix elements for these last three.

Once again the  $Z$ - $\gamma$  mixing has been neglected but it is a simple matter to include it in the manner described previously.

Performing the Laurent expansion in  $p_1^2$  and  $p_2^2$  doubly resonant part corresponding to the process  $e^+e^- \rightarrow (Z^0 \rightarrow f_1\bar{f}_1)(Z^0 \rightarrow f_2\bar{f}_2)$  is

$$\begin{aligned} \mathcal{M} = \sum_i [\bar{v}_{e^+} T_{\mu\nu}^i u_{e^-}] M_i(s, t, u, s_p, s_p) &\times \frac{F_{ZZ}(s_p)}{p_1^2 - s_p} [\bar{u}_{f_1} V_{Zf_1}^\mu(s_p) v_{\bar{f}_1}] \\ &\times \frac{F_{ZZ}(s_p)}{p_2^2 - s_p} [\bar{u}_{f_2} V_{Zf_2}^\mu(s_p) v_{\bar{f}_2}] \end{aligned} \quad (24)$$

in which the final-state fermion currents factorize. Because Eq. (24) is the doubly resonant part of the complete matrix element it must be gauge-invariant. In lowest order it is

$$\mathcal{M} = \sum_{i=1}^2 [\bar{v}_{e^+} T_{\mu\nu}^i u_{e^-}] M_i \cdot \frac{1}{p_1^2 - s_p} [\bar{u}_{f_1} V_{Zf_1}^\mu v_{\bar{f}_1}] \cdot \frac{1}{p_2^2 - s_p} [\bar{u}_{f_2} V_{Zf_2}^\mu v_{\bar{f}_2}] \quad (25)$$

in which  $T_{\mu\nu}^1 M_1 = \gamma_\mu (/p_{e^-} - /p_1) \gamma_\nu /t$ ,  $T_{\mu\nu}^2 M_2 = \gamma_\nu (/p_{e^-} - /p_2) \gamma_\mu /u$ . The final state vertices then take the form  $V_{Zf}^\mu = ie\gamma_\mu (\beta_L^f \gamma_L + \beta_R^f \gamma_R)$ . As usual  $\gamma_L$  and  $\gamma_R$  are the left-

and right-hand helicity projection operators. The left- and right-handed couplings of the  $Z^0$  to a fermion  $f$  are

$$\beta_L^f = \frac{t_3^f - \sin^2 \theta_W Q^f}{\sin \theta_W \cos \theta_W}, \quad \beta_R^f = -\frac{\sin \theta_W Q^f}{\cos \theta_W}.$$

Squaring the matrix element and integrating over the final-state fermion momenta but with fixed invariant mass  $p_1^2$  and  $p_2^2$  gives the differential cross-section for physical  $Z^0$  production

$$\frac{\partial^3 \sigma}{\partial t \partial p_1^2 \partial p_2^2} = \frac{\pi \alpha^2}{s^2} (|\beta_L^e|^4 + |\beta_R^e|^4) \left\{ \frac{t}{u} + \frac{u}{t} + \frac{2(p_1^2 + p_2^2)}{ut} - p_1^2 p_2^2 \left( \frac{1}{t^2} + \frac{1}{u^2} \right) \right\} \rho(p_1) \rho(p_2) \quad (26)$$

with

$$\begin{aligned} \rho(p) &= \frac{\alpha}{6\pi} \sum_f (|\beta_L^f|^2 + |\beta_R^f|^2) \frac{p^2}{|p^2 - s_p|^2} \theta(p_0) \theta(p^2) \\ &\approx \frac{1}{\pi} \frac{p^2 (\Gamma_Z/M_Z)}{(p^2 - M_Z^2)^2 + \Gamma_Z^2 M_Z^2} \theta(p_0) \theta(p^2) \end{aligned}$$

The  $\theta$  functions are generated in the phase space integrations and select the positive energy component. Thus the differential cross-section describes the production of an object that has both finite range and positive energy as would be expected for a physical particle. The function  $\rho(p) \rightarrow \delta(p^2 - M_Z^2) \theta(p_0)$  as  $\Im(s_p) \rightarrow 0$  that is the well-known result obtained by cutting in free propagator. Final state interactions could be built in to cross-section via the convolution kernel  $\rho(p_1) \cdot \rho(p_2)$ .

Integrating over  $t$ ,  $p_1^2$  and  $p_2^2$  gives the total production cross-section for  $e^+ e^- \rightarrow Z^0 Z^0$  gives

$$\sigma(s) = \int_0^s dp_1^2 \int_0^{(\sqrt{s} - \sqrt{p_1^2})^2} dp_2^2 \sigma(s; p_1^2, p_2^2) \rho(p_1^2) \rho(p_2^2), \quad (27)$$

where

$$\sigma(s; p_1^2, p_2^2) = \frac{2\pi \alpha^2}{s^2} (|\beta_L^e|^4 + |\beta_R^e|^4) \left\{ \left( \frac{1 + (p_1^2 + p_2^2)^2/s^2}{1 - (p_1^2 + p_2^2)/s} \right) \ln \left( \frac{-s + p_1^2 + p_2^2 + \lambda}{-s + p_1^2 + p_2^2 - \lambda} \right) - \frac{\lambda}{s} \right\}$$

and  $\lambda = \sqrt{s^2 + p_1^4 + p_2^4 - 2sp_1^2 - 2sp_2^2 - 2p_1^2 p_2^2}$ . Setting  $p_1^2 = p_2^2 = M_Z^2$  reproduces the results of Brown and Mikaelian[35].

The result given here is superficially rather similar to those of other authors[36, 37]. The difference lies in the fact in the present case strictly only the doubly resonant part of the matrix element has been included and manifests itself in a change in the convolution kernel  $\rho$ . In higher orders the approach of ref.s [36] and [37] will lead to the evaluation of off-shell matrix elements that will generally be gauge-dependent. In the method described here the form factors in the matrix element are evaluated at the pole and are therefore gauge-invariant. This also simplifies the evaluation of

the integrals like those in Eq. (27) as the higher-order form-factors are not integrated over. Only factors deriving from the kinematic tensors are integrated.

After the  $Z^0$ 's have propagated and decayed the final state fermions so-produced can interact. Because the  $Z^0$ 's have propagated a finite distance from their point of production, the final-state interactions are expected to be suppressed. This is generally found to be the case[38, 39, 40]. Such a suppression was already observed[41, 42] in the case of the  $e^+e^- \rightarrow Z^0 \rightarrow f\bar{f}$  where the exact correction to the cross-section from the interference of initial- and final-state interference was shown to vanish near resonance and may be understood as suppression due the finite range propagated by the  $Z^0$ .

## 6 Conclusions

In the foregoing paper it has been shown how to use the Laurent expansion to generate finite-order matrix elements, for processes involving unstable particles, that are exactly gauge-invariant. This is achieved without the *ad hoc* introduction of finite widths and without the need for gauge-invariant self-energies or vertex corrections. Indeed, because of the interpretation of the resonant part as being the finite range interaction of an unstable particle being produced, propagating and decaying, the Laurent expansion should be carried out in all calculational schemes in order to preserve the separation of the matrix element into physically distinguishable parts.

The approach was also used in obtaining physically meaningful expressions for the production cross-sections for the production cross-sections for unstable particles.

## References

- [1] A. Pilaftsis, *Z. Phys. C* **47** (1990) 95.
- [2] G. Eilam, J. L. Hewett and A. Soni, *Phys. Rev. Lett.* **67** (1991) 1979.
- [3] R. G. Stuart, in *Z<sup>0</sup> Physics; Proceedings of the XXVth Rencontre de Moriond*, ed. J. Tran Thanh Van, Editions Frontières, Gif-sur-Yvette (1990) p.41
- [4] M. Nowakowski and A. Pilaftsis, *Z. Phys. C* **60** (1993) 121.
- [5] U. Baur and D. Zeppenfeld, Madison preprint MAD/PH/878, Bulletin Board hep-ph 9503304.
- [6] C. G. Papadopoulos, CERN preprint CERN-TH/95-46, Bulletin Board hep-ph 9503276.
- [7] A. Aeppli, F. Cuypers and G. J. van Oldenborgh, *Phys. Lett. B* **314** (1993) 413.
- [8] A. Aeppli, G. J. van Oldenborgh and D. Wyler, *Nucl. Phys. B* **428** (1994) 126.

- [9] R. G. Stuart, *Phys. Lett.* **B 262** (1991) 113.
- [10] R. G. Stuart, *Phys. Lett.* **B 272** (1991) 353.
- [11] R. G. Stuart, *Phys. Rev. Lett.* **70** (1993) 3193.
- [12] R. G. Stuart, Michigan preprint UM-TH-95-06, Bulletin Board hep-ph 9504215.
- [13] A. Denner, G. Weiglein and S. Dittmaier, *Phys. Lett.* **B 333** (1994) 420; Bielefeld preprint BI-TP-94-50, Bulletin Board hep-ph 9410338.
- [14] R. J. Eden, P. V. Landshoff, D. I. Olive and J. C. Polkinghorne, *The Analytic S-Matrix*, Cambridge University Press, Cambridge (1966).
- [15] H. P. Stapp, *Nuovo Cimento* **32** (1964) 103.
- [16] J. Gunson, *J. Math. Phys.* **6** (1965) 827; **6** (1965) 845; **6** (1965) 852.
- [17] J. Schwinger, *Ann. Phys.* **9** (1960) 169.
- [18] M. Veltman, *Physica* **29** (1963) 186.
- [19] A. Martin, in *Z<sup>0</sup> Physics 1990; NATO Advanced Study Institute Cargese Summer School* NATO ASI (1990) p.483.
- [20] D. C. Kennedy and B. W. Lynn, *Nucl. Phys.* **B 322** (1989) 1.
- [21] D. C. Kennedy, B. W. Lynn, C. J.-C. Im and R. G. Stuart, *Nucl. Phys.* **B 321** (1989) 83.
- [22] G. Degrossi and A. Sirlin, *Phys. Rev.* **D 46** (1992) 3104.
- [23] L. Baulieu and R. Coquereaux, *Ann. Phys.* **140** (1982) 163
- [24] H. Veltman, *Z. Phys.* **C 62** (1994) 35.
- [25] A. Sirlin, *Phys. Rev. Lett.* **67** (1991) 2127.
- [26] A. Sirlin, *Phys. Lett.* **B 267** (1991) 240.
- [27] R. E. Peierls, *Proceedings of the 1954 Glasgow Conference on Nuclear and Meson Physics*, Pergamon Press, New York, (1955) 296.
- [28] M. Lévy, *Nuovo Cimento* **13** (1959) 115.
- [29] S. Willenbrock and G. Valencia, *Phys. Lett.* **B 259** (1991) 373.
- [30] T. Bhattacharya and S. Willenbrock, Brookhaven preprint, BNL-56481
- [31] V. S. Fadin, V. A. Khoze and A. D. Martin, *Phys. Lett.* **B 311** (1993) 403.

- [32] D. Bardin, W. Beenakker and A. Denner, *Phys. Lett.* **B 317** (1993) 213.
- [33] V. S. Fadin, V. A. Khoze, A. D. Martin and A. Chapovsky, Durham preprint DTP/94/116, Bulletin board hep-ph 9501214.
- [34] W. A. Bardeen and W.-K. Tung, *Phys. Rev.* **173** (1968) 1423.
- [35] R. W. Brown and K. O. Mikaelian, *Phys. Rev.* **D 19** (1979) 922.
- [36] T. Muta, R. Najima and S. Wakaizumi, *Mod. Phys. Lett.* **A1** (1986) 203.
- [37] A. Denner and T. Sack, *Z. Phys.* **C 45** (1990) 439.
- [38] K. Melnikov and O. Yakovlev, *Phys. Lett.* **B 324** (1994) 217;
- [39] K. Melnikov and O. Yakovlev, Mainz preprint MZ-TH/95-01, Bulletin Board hep-ph 9501358.
- [40] V. A. Khoze, Durham preprint DTP-94-114, Bulletin Board hep-ph 9412239.
- [41] J. H. Kühn and R. G. Stuart, *Phys. Lett.* **B 200** (1988) 360;
- [42] J. H. Kühn, S. Jadach, R. G. Stuart and Z. Wąs, *Z. Phys.* **C 38** (1988) 609; E *ibid.* **C 45** (1990) 528;